Graded Adjoint Logic

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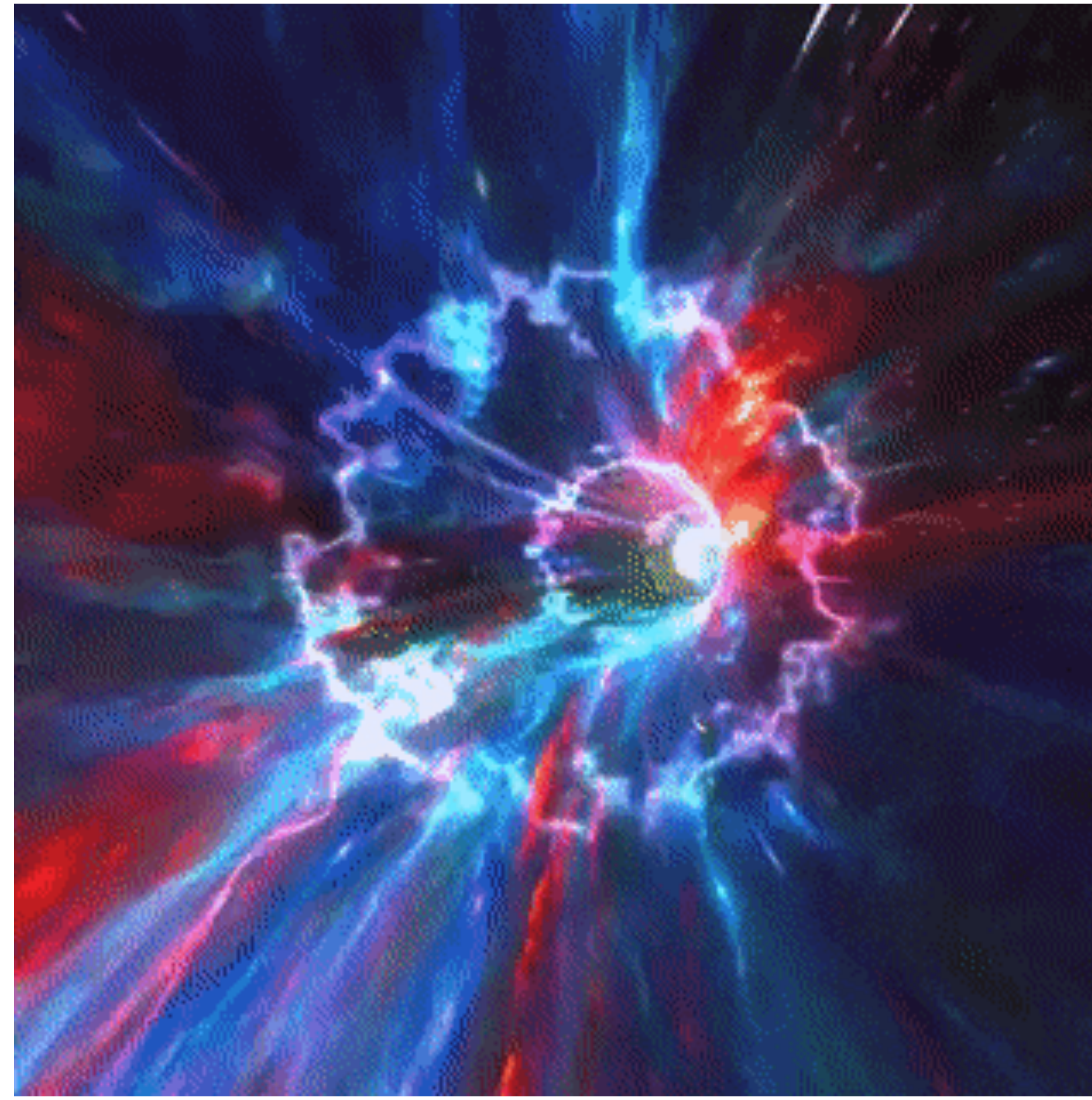


Substructural Logics

Logics that limit how we are allowed to use our hypotheses.

Substructural Logics

Logics that limit which structural rules are present, or how they are used.

Substructural Logics

The strucrural rules:

$$\frac{\Gamma_1, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ \textbf{weak} \qquad \frac{\Gamma_1, A, A, \Gamma_2 \vdash B}{\Gamma_1, A, \Gamma_2 \vdash B}\ \textbf{cont}$$

Substructural Logics

Intuitionistic logic:

- Weakening
- Contraction

Linear logic:

- ~~Weakening~~
- ~~Contraction~~

Affine logic:

- Weakening
- ~~Contraction~~

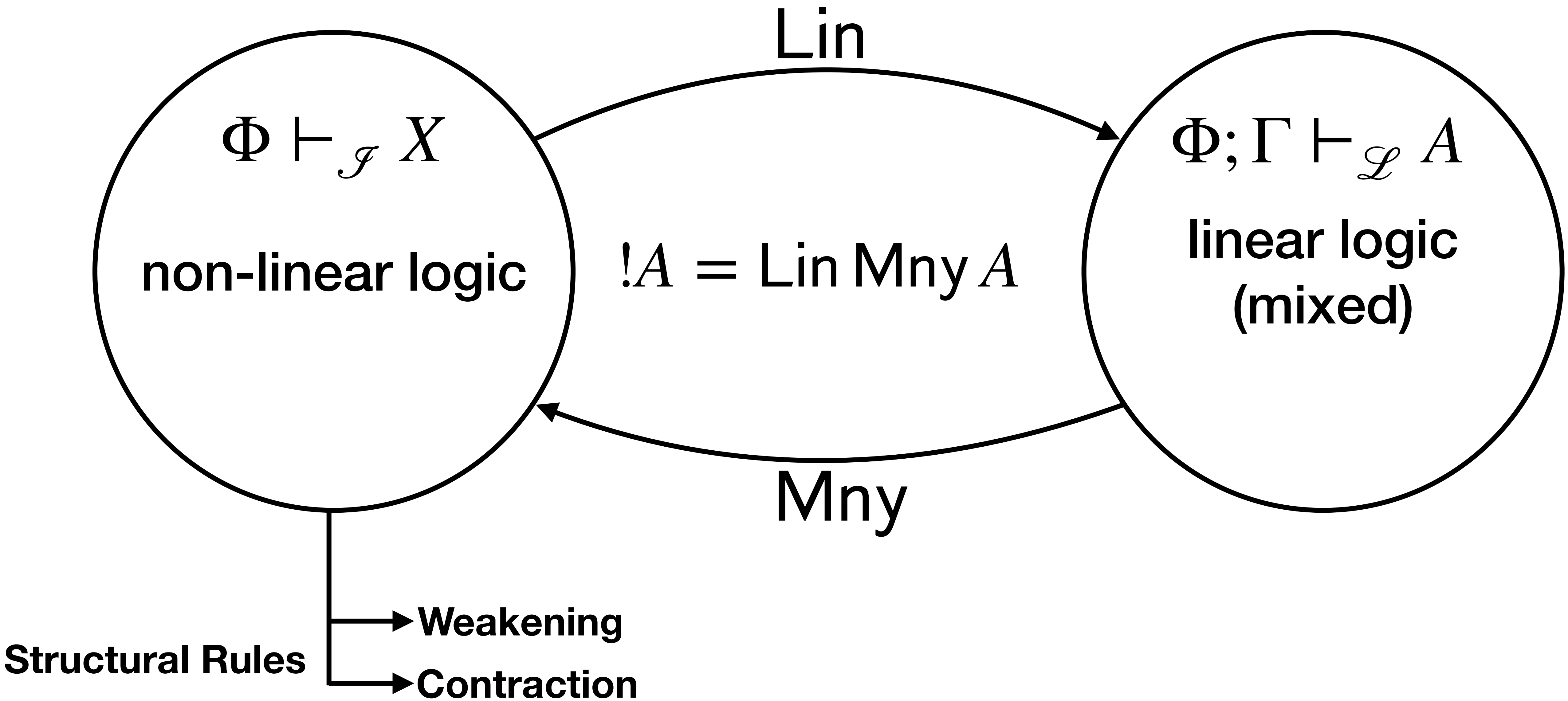
Relevant logic:

- ~~Weakening~~
- Contraction

Substructural Logics

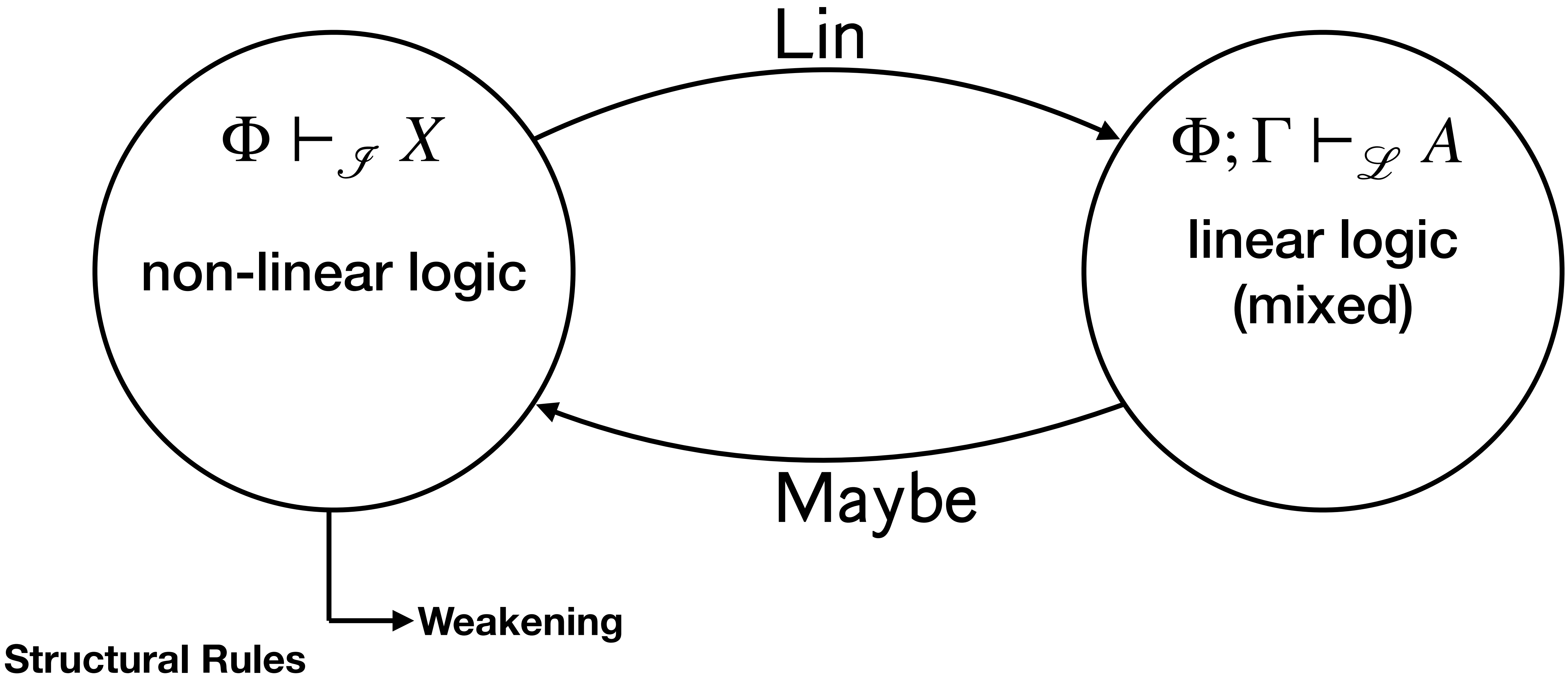
Is there a framework that can be instantiated to the various substructural logics?

Linear/Non-Linear (LNL) Logic

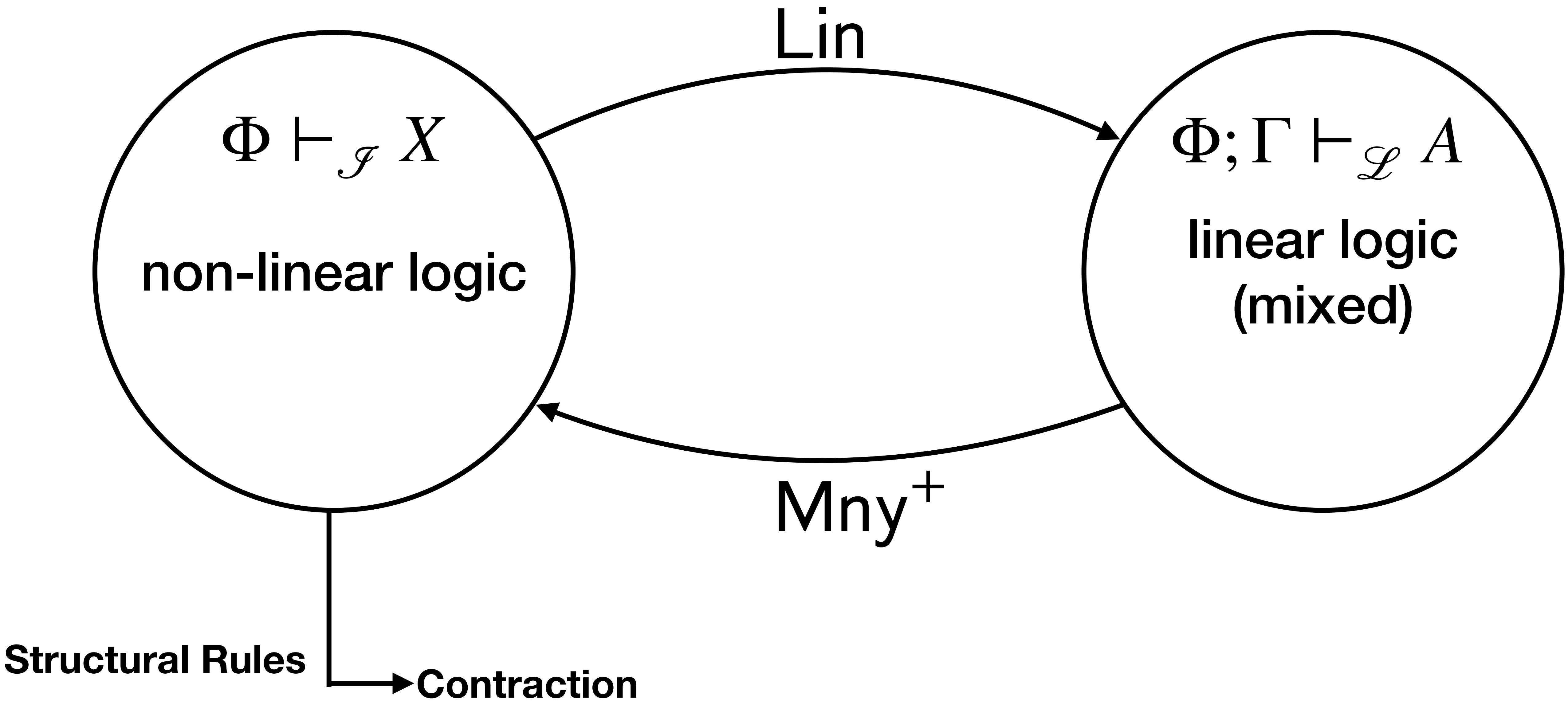


A Mixed Linear and Non-Linear Logic: Proofs, Terms and Models
Nick Benton
https://www.cl.cam.ac.uk/techreports/UCAM-CL-TR-352.html

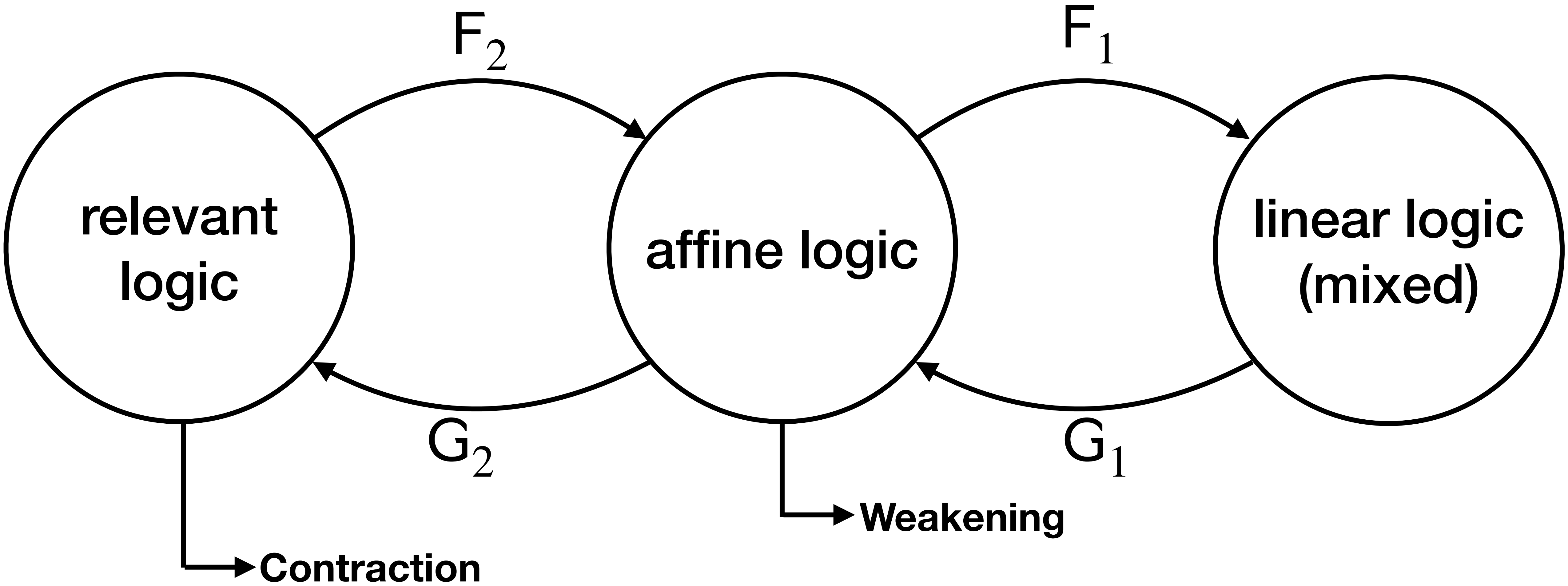
Linear/Affine Logic



Linear/Relevant Logic



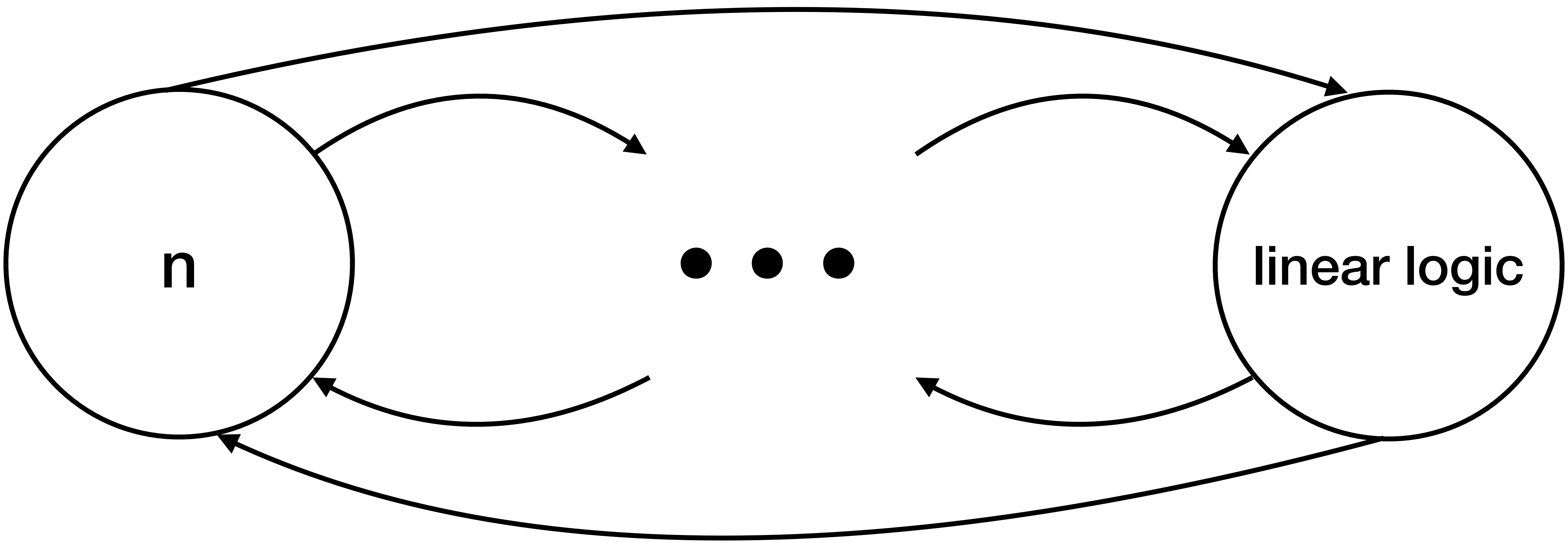
Combining Structural Rules



$$!A = F_1 F_2 G_2 G_1 A$$

Comparing hierarchies of types in models of linear logic
Paul-André Melliés
http://dx.doi.org/10.1016/j.ic.2003.10.003

Families of Modalities through Adjunctions



Adjoint Logic

Modes : $(\mathscr{M}, \geq)$

Structural Properties : $\sigma : \mathscr{M} \rightarrow \mathscr{P}(\{\mathsf{W}, \mathsf{C}, \mathsf{E}\})$

Judgments : $A^1_{m_1}, \ldots, A^n_{m_n} \vdash B_m$**, with** $m_i \geq k$

Modalities : $\uparrow^m_k A_k$**, with** $m \geq k$

$\downarrow^k_m A_k$**, with** $k \geq m$

Adjoint Logic
Pruiksma et al.
https://www.cs.cmu.edu/~fp/papers/adjoint18b.pdf

Adjoint Logic

Modes control <u>which</u> structural rules are present.

$$\frac{\mathsf{W} \in \sigma(m) \quad \Gamma_1, \Gamma_2 \vdash B_k}{\Gamma_1, A_m, \Gamma_2 \vdash B_k}\ \textbf{weak} \qquad \frac{\Gamma_2 \geq m \geq k \quad \begin{array}{c}\Gamma_2 \vdash A_m \\ \Gamma_1, A_m, \Gamma_3 \vdash B_k\end{array}}{\Gamma_1, A_m, \Gamma_3 \vdash B_k}\ \textbf{cut}$$

$$\frac{\mathsf{C} \in \sigma(m) \quad \Gamma_1, A_m, A_m, \Gamma_2 \vdash B_k}{\Gamma_1, A_m, \Gamma_2 \vdash B_k}\ \textbf{cont}$$

<u>Adjoint Logic</u>
Pruiksma et al.
<u>https://www.cs.cmu.edu/~fp/papers/adjoint18b.pdf</u>

Graded Linear Logic

Grades : $(\mathscr{R}, \mathrm{m}, \circledast, \mathrm{a}, \oplus, \leq)$

Judgments : $r_1 \odot A_1, \ldots, r_n \odot A_n \vdash B$

Modalities : $\Box_r A$

Graded Linear Logic

$$(\mathscr{R}, \mathsf{m}, \circledast, \mathsf{a}, \oplus, \leq)$$

$$\cdots$$

$$(r_1, \ldots, r_n) \odot (A_1, \ldots, A_n) \vdash B$$

$$\Updownarrow$$

$$r_1 \odot A_1, \ldots, r_n \odot A_n \vdash B$$

Graded Linear Logic

Grades control how structural rules used.

$$\frac{(\gamma_1, \gamma_2) \odot (\Gamma_1, \Gamma_2) \vdash B}{(\gamma_1, \mathsf{a}, \gamma_2) \odot (\Gamma_1, A, \Gamma_2) \vdash B}\ \textbf{weak}$$

$$\frac{\gamma_2 \odot \Gamma_2 \vdash A \qquad (\gamma_1, r, \gamma_3) \odot (\Gamma_1, A, \Gamma_3) \vdash B}{(\gamma_1, r \circledast \gamma_2, \gamma_3) \odot (\Gamma_1, \Gamma_2, \Gamma_3) \vdash B}\ \textbf{cut}$$

$$\frac{(\gamma_1, r_1, r_2, \gamma_2) \odot (\Gamma_1, A, A, \Gamma_2) \vdash B}{(\gamma_1, (r_1 \oplus r_2), \gamma_2) \odot (\Gamma_1, A, \Gamma_2) \vdash B}\ \textbf{cont}$$

Graded Adjoint Logic

Brings these two similar, but different ideas together.

But, first, let's go back to the beginning....

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

What's the category of coalgebras?

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1.What's the category of coalgebras?

What're the free coalgebras?

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1.What's the category of coalgebras?

Q1.2 What're the free coalgebras?

A1.2 Pairs $(mA, \delta^m A)$ **where** $\delta^m : mA \to m^2 A$

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1.What's the category of coalgebras?

Q1.2 What're the free coalgebras?

A1.2 Pairs $(mA, \delta^m A)$ **where** $\delta^m : mA \to m^2 A$

Let's abstract these!

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1: What's the category of coalgebras?

Q1.2: What're the free coalgebras?

A1.2: Pairs $(mA, \delta^m A)$ **where** $\delta^m : mA \to m^2 A$

A1: Pairs (A, h_A) **where** $h_A : A \to mA$

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1: What's the category of coalgebras?

Q1.2: What're the free coalgebras?

A1.2: Pairs $(mA, \delta^m A)$ **where** $\delta^m : mA \to m^2 A$

A1: Pairs (A, h_A) **where** $h_A : A \to mA$

Why is this important?

Comonads from Adjunctions

Comonad: $(m, \delta^m, \varepsilon^m)$

Q1: What's the category of coalgebras?

Q1.2: What're the free coalgebras?

A1.2: Pairs $(mA, \delta^m A)$ **where** $\delta^m : mA \to m^2 A$

A1: Pairs (A, h_A) **where** $h_A : A \to mA$

The category of coalgebras is endowed with the structure of the comonad!

Comonads from Adjunctions

The category of coalgebras is endowed with the structure of the comonad!

$$(m, \delta^m, \varepsilon^m) = (!, \delta^!, \varepsilon^!)$$

$$\Updownarrow$$

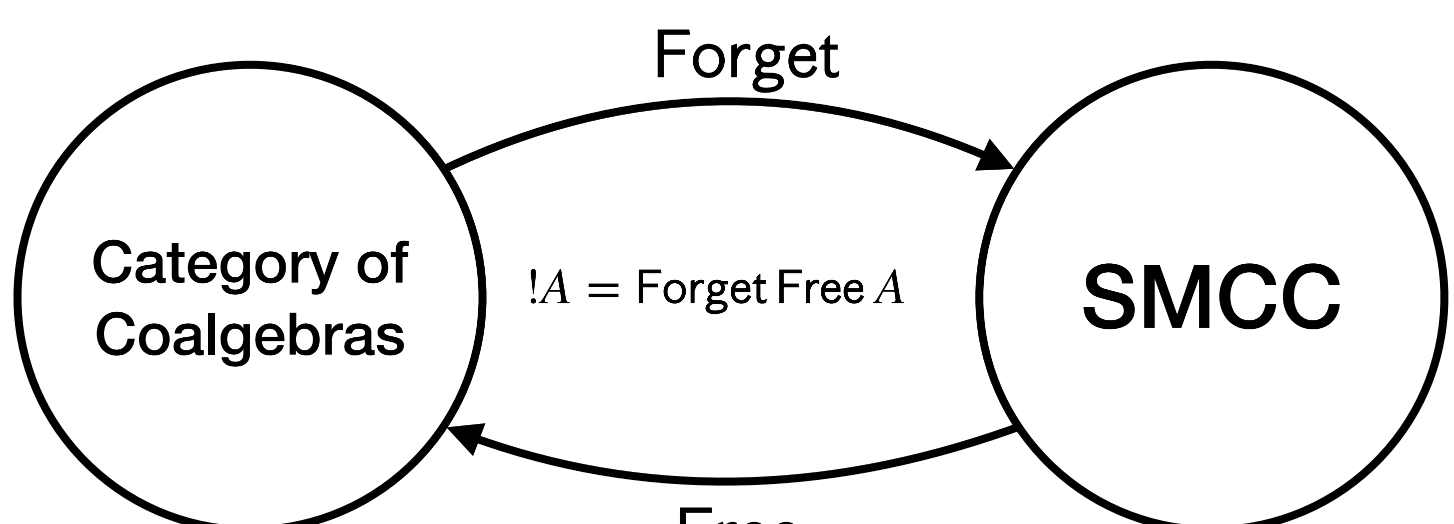
The category of coalgebras is cartesian!

Comonads from Adjunctions

The category of coalgebras is related to the original category through an adjunction!

$$(m, \delta^m, \varepsilon^m) = (!, \delta^!, \varepsilon^!)$$

$$\Updownarrow$$

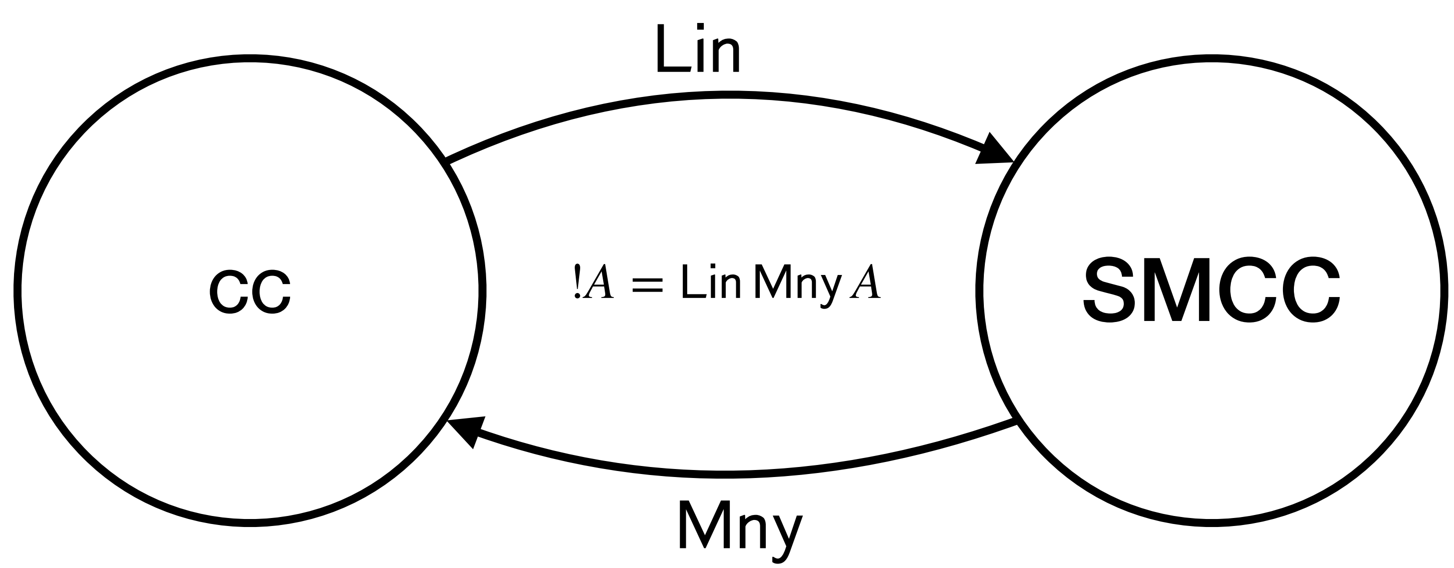


LNL Models

Benton abstracted the category of coalgebras!

$$(m, \delta^m, \varepsilon^m) = (!, \delta^!, \varepsilon^!)$$

$$\Updownarrow$$



What's the story for Graded Necessity Modalities?

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

$$\delta_{r_1,r_2} : \Box_{r_1 \circledast r_2} A \to \Box_{r_1} \Box_{r_2} A$$

$$\varepsilon : \Box_{mid} A \to A$$

Graded Comonads from Adjunctions

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Q1: What's the category of graded coalgebras?

Graded Comonads from Adjunctions

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Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

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$\varepsilon : \Box_{mid} A \to A$

Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

Pairs $(\Box_r A, \delta_{r,s} A)$ **where**

$$\delta : \Box_{r \circledast s} A \to \Box_r \Box_s A$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

Pairs $(\Box_r A, \delta_{r,s} A)$ **where**

$$\delta : \Box_{r \circledast -} A \to \Box_r \Box_- A$$

Graded Comonads from Adjunctions

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Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

Pairs $(\Box_r A, \delta_{r,\_} A)$ **where**

$$\delta : \Box_{r \circledast \_} A \to \Box_r \Box_\_ A$$

Let's abstract these!

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

$\delta_{r_1,r_2} : \Box_{r_1 \circledast r_2} A \to \Box_{r_1} \Box_{r_2} A$

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Q1: What's the category of graded coalgebras?

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Pairs $(\Box_r A, \delta_{r,\_} A)$ **where**

$$\delta : \Box_{r \circledast \_} A \to \Box_r \Box_\_ A$$

$$\Downarrow$$

$$\hat{A} = \Box_\_ A : \mathscr{R} \to \mathscr{M}$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

$\delta_{r_1,r_2} : \Box_{r_1 \circledast r_2} A \to \Box_{r_1} \Box_{r_2} A$

$\varepsilon : \Box_{mid} A \to A$

Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

$$\hat{A} = \Box_{\_} A : \mathscr{R} \to \mathscr{M}$$

Pairs $(\hat{A}, \delta_{\_,r})$ **where**

$$\delta : \hat{A}(r \circledast s) \to \Box_r \hat{A}(s)$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

$\delta_{r_1,r_2} : \Box_{r_1 \circledast r_2} A \to \Box_{r_1} \Box_{r_2} A$

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Q1: What's the category of graded coalgebras?

Q1.2: What are the free graded coalgebras?

$A : \mathscr{R} \to \mathscr{M}$

Pairs (A, h) **where**

$h_{r,s} : A(r \circledast s) \to \Box_r A(s)$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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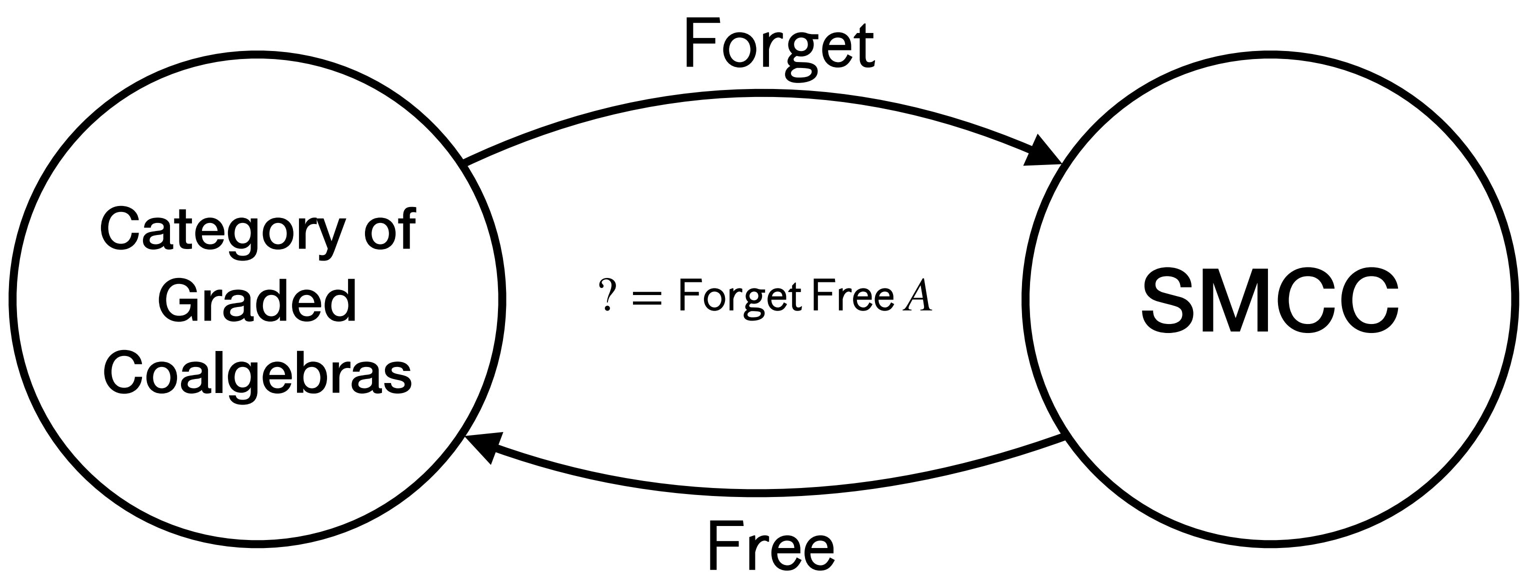

$$\mathsf{Forget}(A, h) = A(\mathsf{mid})$$

$$\mathsf{Free}(A) = (\lambda s\,.\,\Box_s A, \delta)$$

Graded Comonads from Adjunctions

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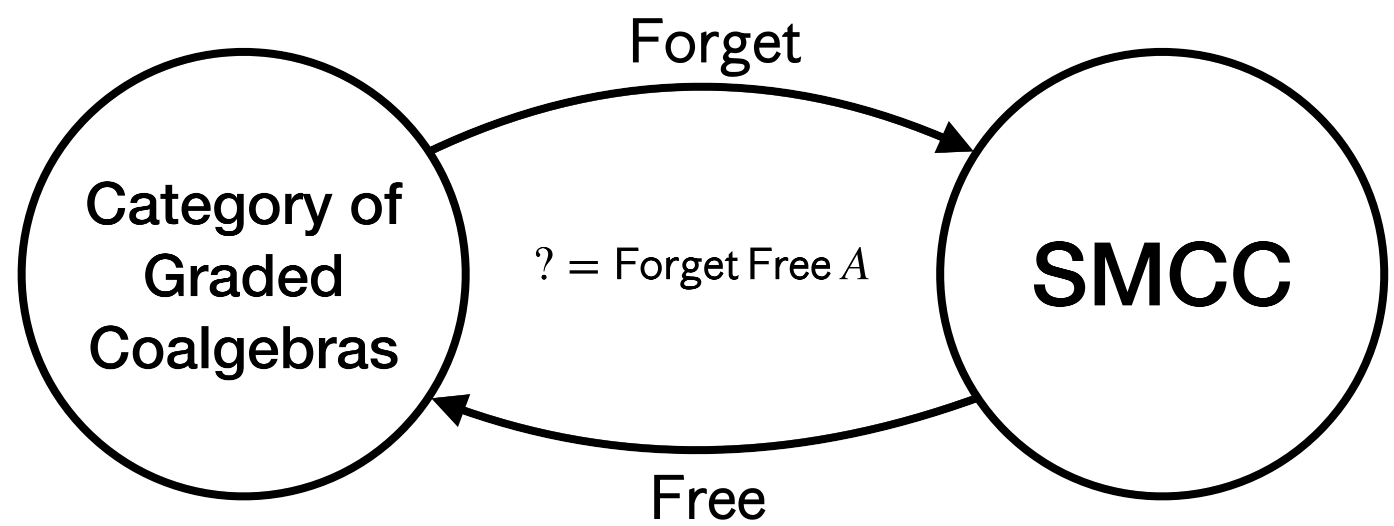

$$\mathsf{Forget}(\mathsf{Free}\, A) = \mathsf{Forget}(\lambda r . \Box_r A, \delta) = \Box_{\mathsf{mid}} A$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

$\delta_{r_1,r_2} : \Box_{r_1 \circledast r_2} A \to \Box_{r_1} \Box_{r_2} A$

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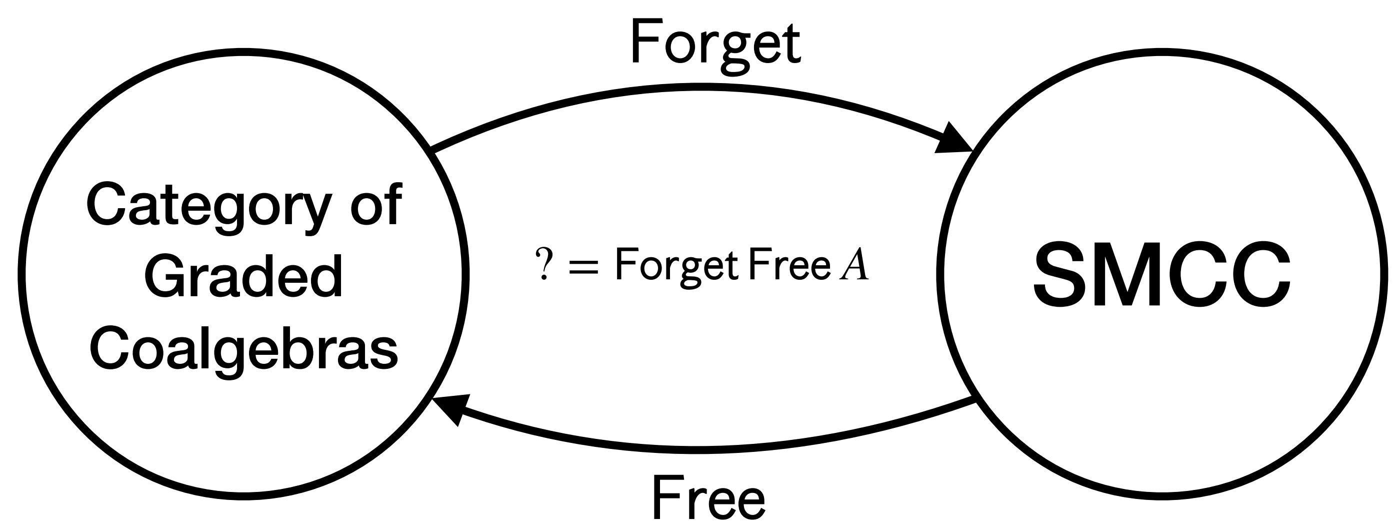

$$\text{Forget}(\text{Free}\, A) = \text{Forget}(\lambda r . \Box_r A, \delta) = \Box_{\text{mid}} A$$

All is not lost!

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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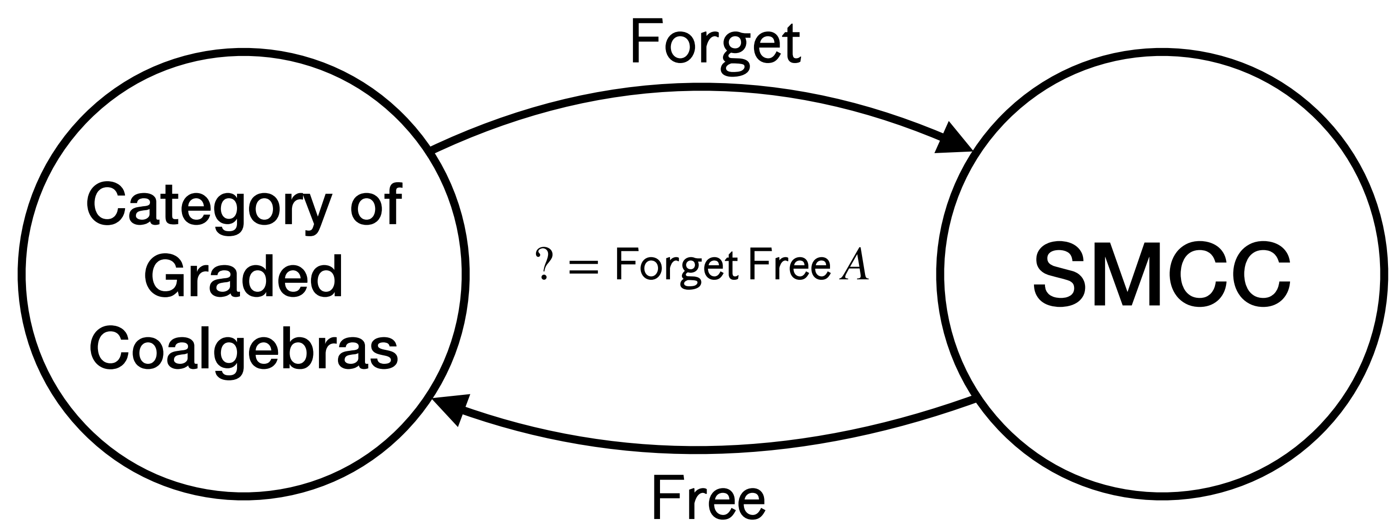
Forget$(A, h) = A$(mid)

Free$(A) = (\lambda r . \Box_r A, \delta)$



We have a graded action:

$$r \odot (A, h) = (\lambda s . A(r \circledast s), \lambda s . h_{r, r \circledast s}) : \mathscr{R} \times \mathscr{M}^{\Box} \to \mathscr{M}^{\Box}$$

where

$$h_{r, r \circledast s} : A(r \circledast (r \circledast s)) \to \Box_r A(r \circledast s)$$

Graded Comonads from Adjunctions

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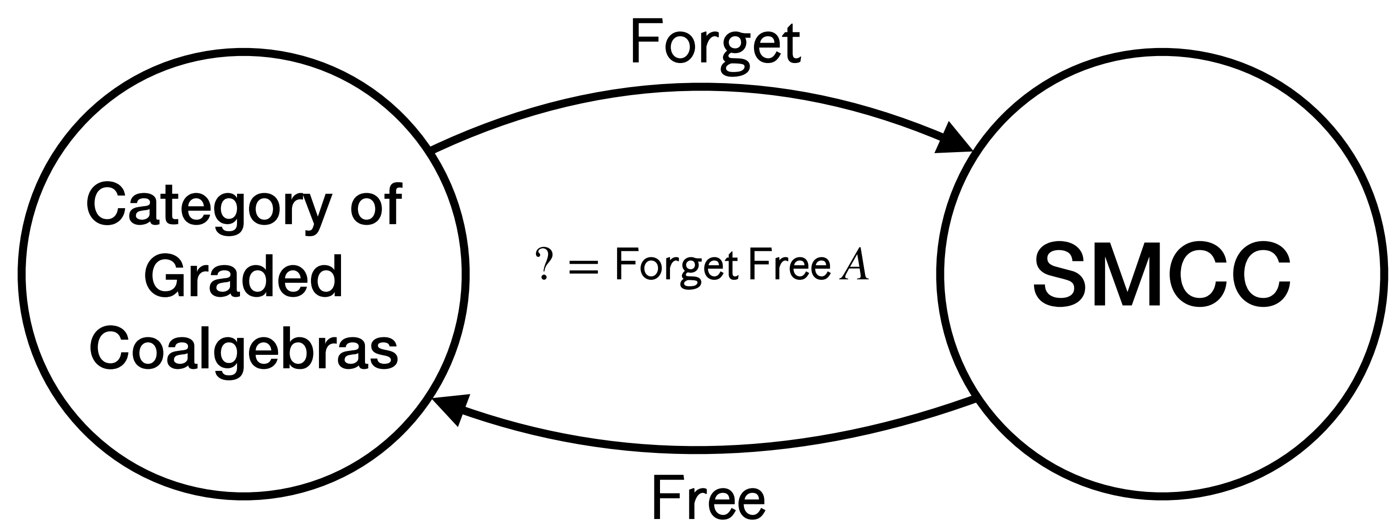

$$\text{Forget}(r \odot \text{Free}\, A) = \text{Forget}(r \odot (\lambda s . \Box_s A, \delta))$$

Graded Comonads from Adjunctions

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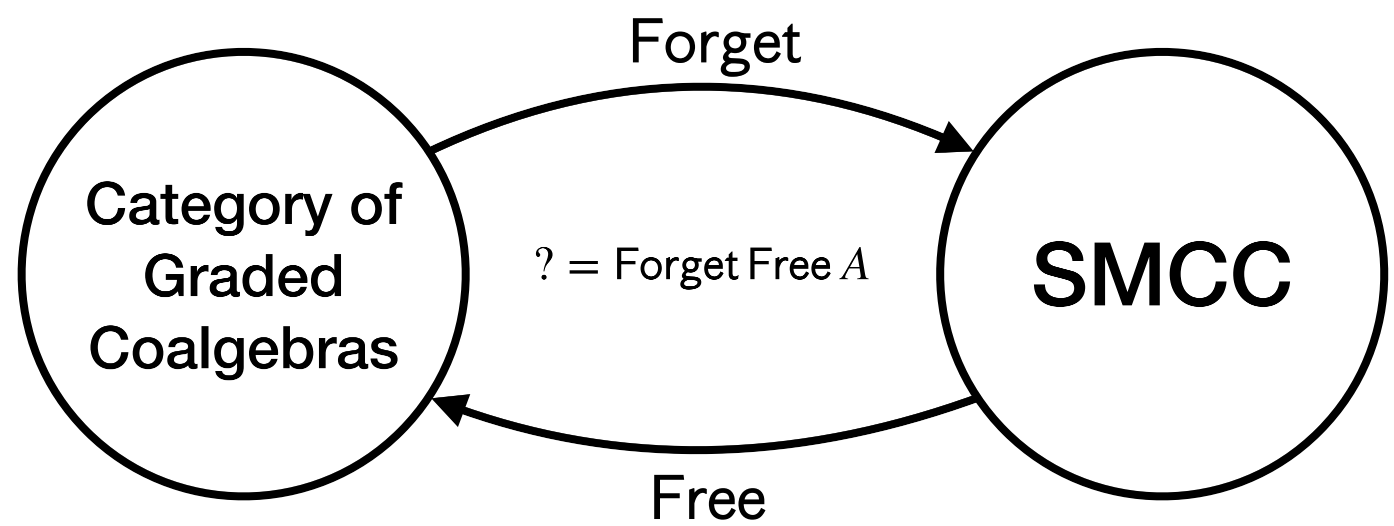
Forget$(A, h) = A$(mid)

Free$(A) = (\lambda r . \Box_r A, \delta)$



$$\text{Forget}(r \odot \text{Free}\, A) = \text{Forget}(r \odot (\lambda s . \Box_s A, \delta))$$

$$= \text{Forget}(\lambda s . \Box_{r \circledast s} A, \lambda s . \delta_{r, r \circledast s})$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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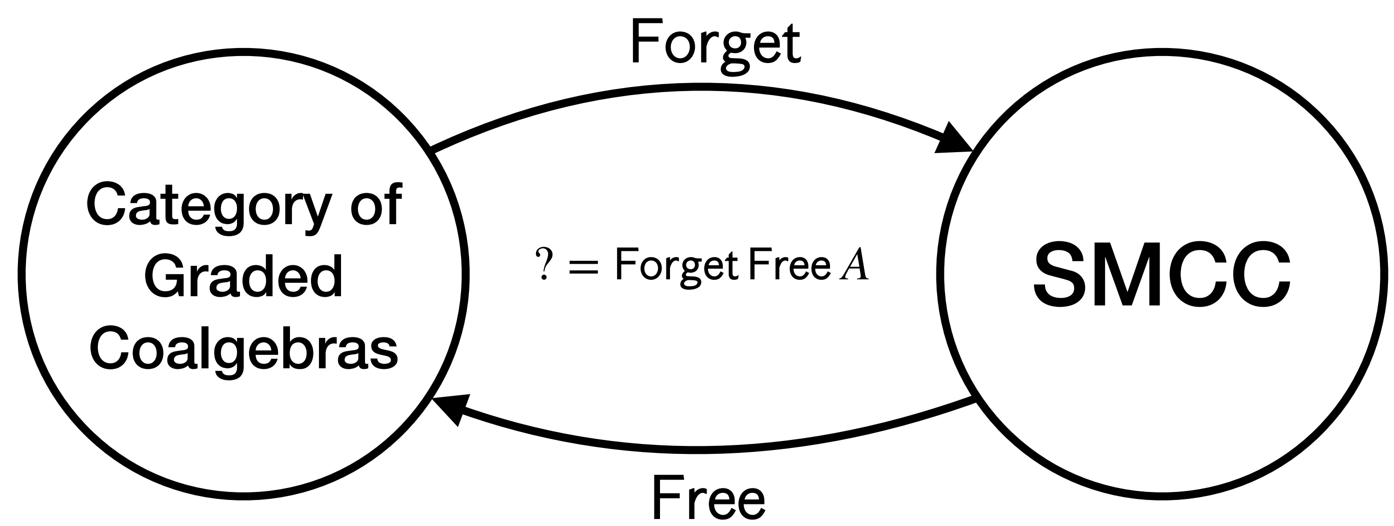
$\text{Forget}(A, h) = A(\text{mid})$

$\text{Free}(A) = (\lambda r . \Box_r A, \delta)$



$$\text{Forget}(r \odot \text{Free}\, A) = \text{Forget}(r \odot (\lambda s . \Box_s A, \delta))$$

$$= \text{Forget}(\lambda s . \Box_{r \circledast s} A, \lambda s . \delta_{r, r \circledast s})$$

$$= \Box_{r \circledast \text{mid}} A$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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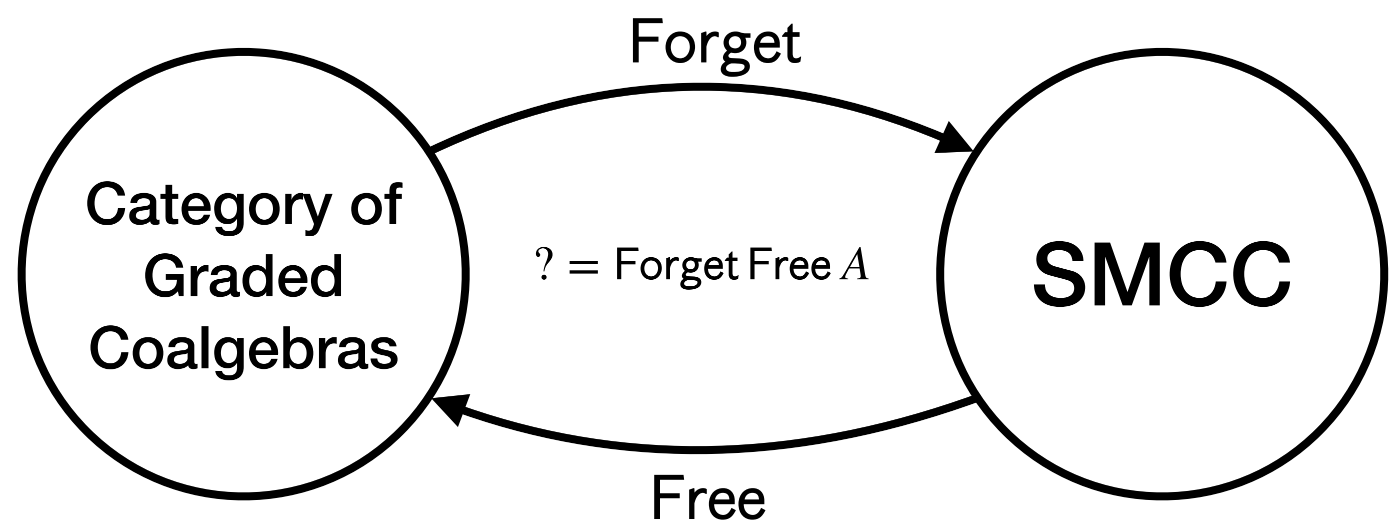

$$\mathsf{Forget}(r \odot \mathsf{Free}\, A) = \mathsf{Forget}(r \odot (\lambda s . \Box_s A, \delta))$$
$$= \mathsf{Forget}(\lambda s . \Box_{r \circledast s} A, \lambda s . \delta_{r, r \circledast s})$$
$$= \Box_{r \circledast \mathsf{mid}} A$$
$$= \Box_r A$$

Graded Comonads from Adjunctions

Graded Comonad: $(\Box_r, \delta_{r_1,r_2}, \varepsilon)$ **where**

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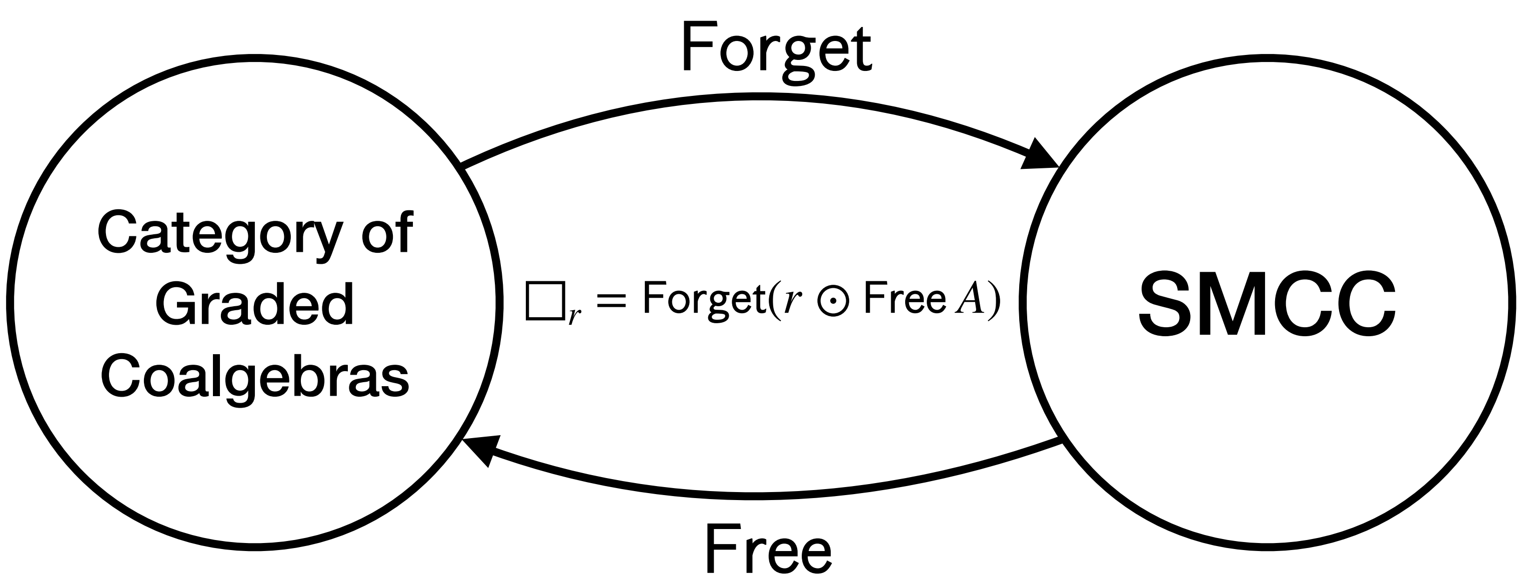
A Double Category Theoretic Analysis of Grad
Shin-ya Katsumata



$$\text{Forget}(A, h) = A(\text{mid})$$

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$$r \odot (A, h) = (\lambda s . A(r \circledast s), \lambda s . h_{r, r \circledast s}) : \mathscr{R} \times \mathscr{M}^{\Box} \to \mathscr{M}^{\Box}$$

Graded Comonads from Adjunctions

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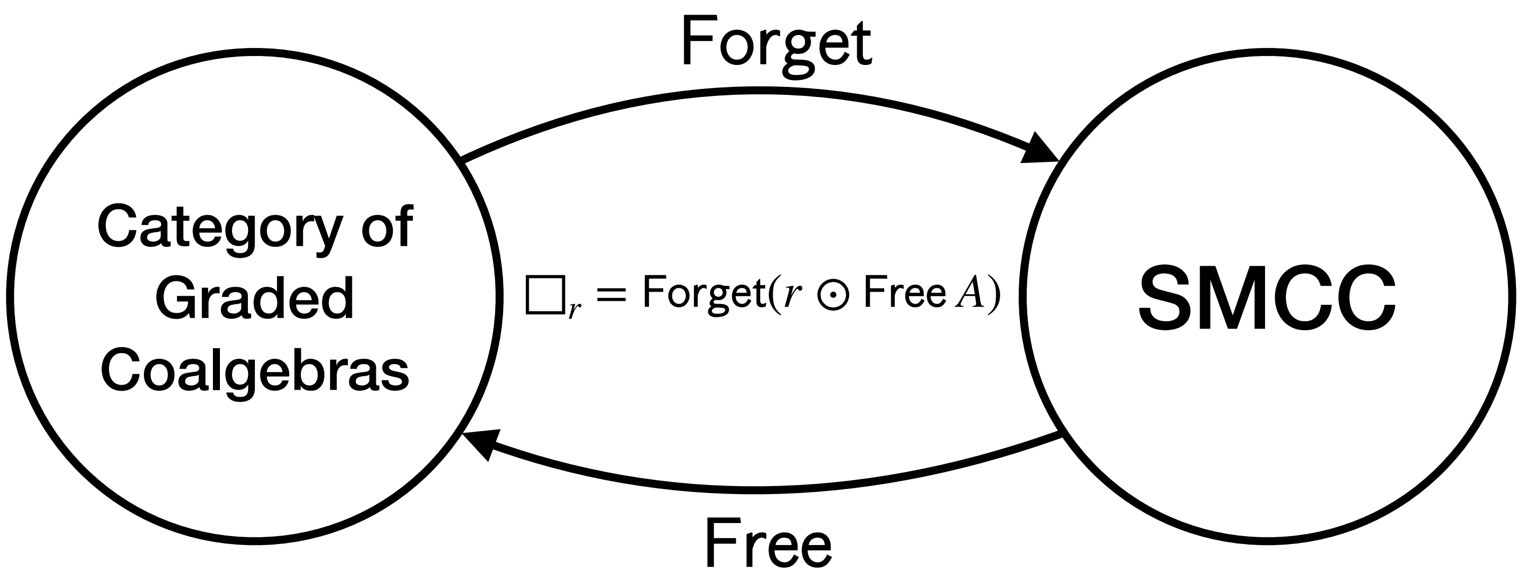

A Double Category Theoretic Analysis of Graded Linear Exponential Comonads
Shin-ya Katsumata
https://link.springer.com/chapter/10.1007/978-3-319-89366-2_6

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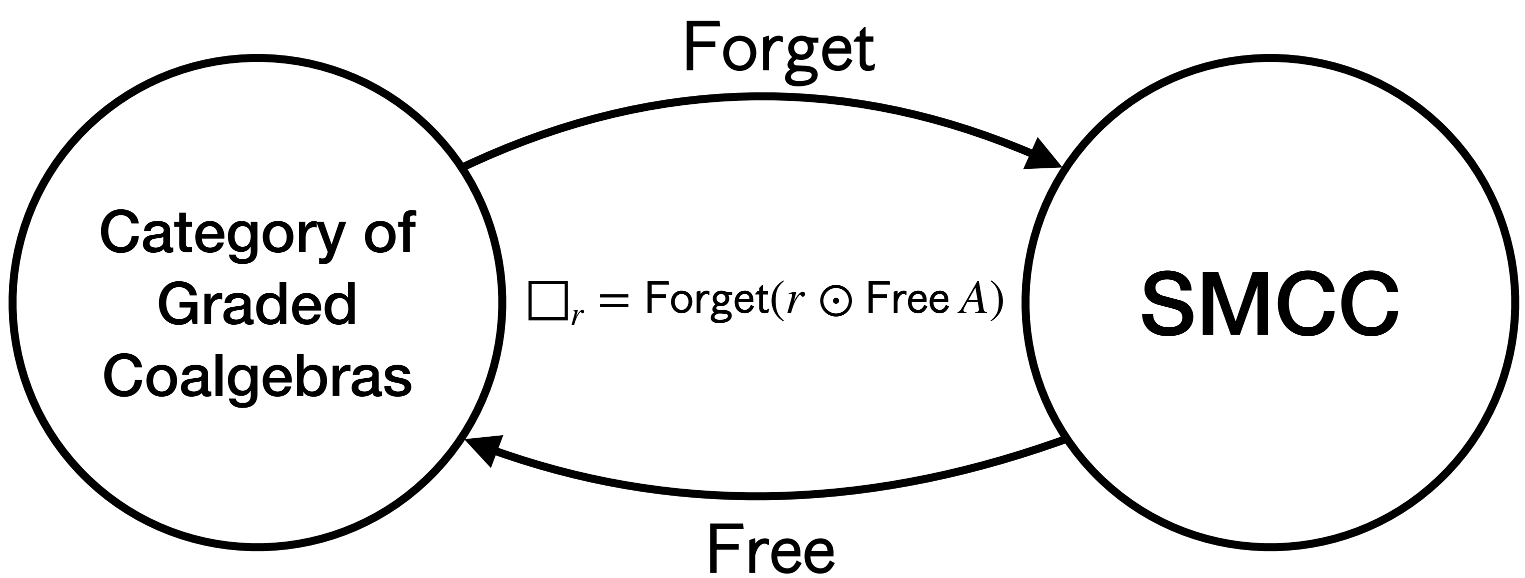
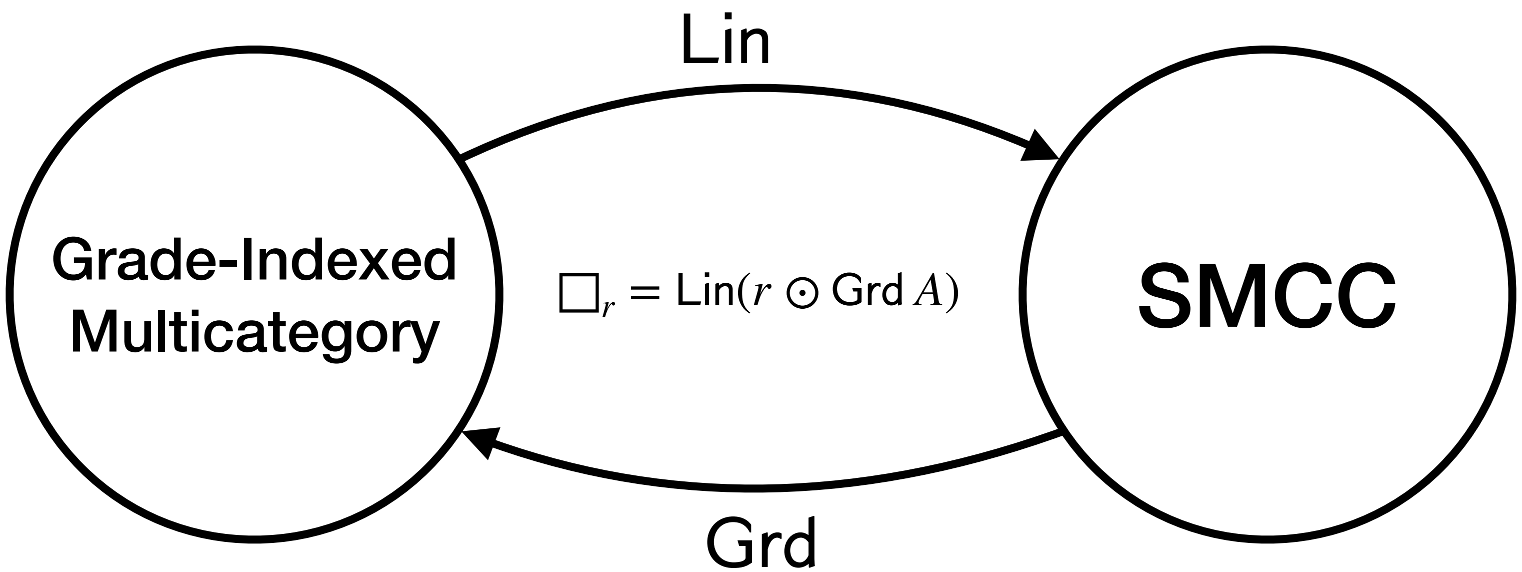


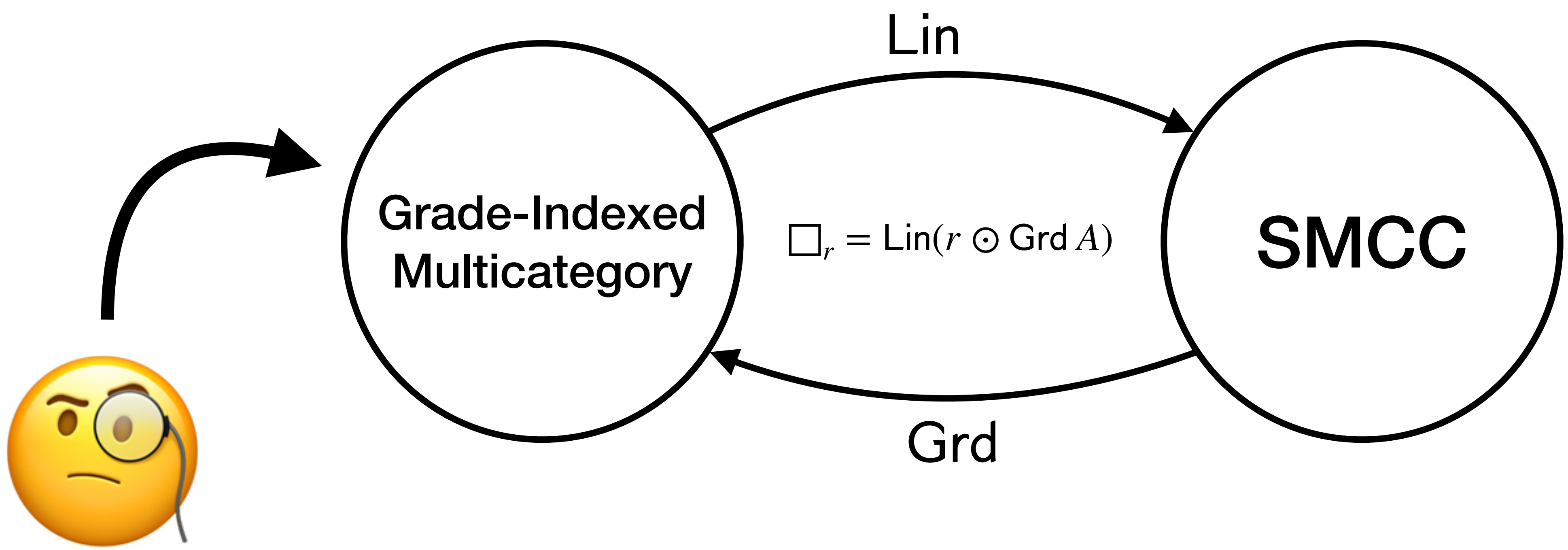
What about abstracting this in the style of Benton?

Mixed Graded/Linear Logic



$$r \odot A_s = A_{r \circledast s} : \mathscr{R} \times \mathscr{G} \to \mathscr{G}$$

Mixed Graded/Linear Logic



$$r \odot A_s = A_{r \circledast s} : \mathscr{R} \times \mathscr{G} \to \mathscr{G}$$

Grade Indexed Multicategory

Suppose $\mathscr{M}$ **is a SMC and**
$(\mathscr{R}, \mathrm{mid}, \circledast, \leq)$ **is a preordered monoid.**

Objects: pairs A_r **for** $A \in \mathrm{Obj}(\mathscr{M}), r \in \mathscr{R}$

Morphisms:

$$f : \langle A^1_{r_1}, \ldots, A^n_{r_n} \rangle \to B_s$$

Identity:

$$\mathrm{id} : \langle A_r \rangle \to A_r$$

Acting:

$$\mathrm{act}_r : \mathrm{Hom}(A^1_{r_1}, \ldots, A^n_{r_n}; B_s) \to \mathrm{Hom}(A^1_{r \circledast r_1}, \ldots, A^n_{r \circledast r_n}; B_{r \circledast s})$$

Graded Multicategory

Suppose $\mathscr{M}$ **is a SMC and**
$(\mathscr{R}, \mathsf{mid}, \circledast, \leq)$ **is a preordered monoid.**

Objects: A **for** $A \in \mathsf{Obj}(\mathscr{M})$

Morphisms:

$$f : \langle A^1_{r_1}, \ldots, A^n_{r_n} \rangle \to B$$

Identity:

$$\mathsf{id} : \langle A_{\mathsf{mid}} \rangle \to A_{\mathsf{mid}}$$

Graded Multicategory

Composition:

Given:

$$f_1 : \langle X^{11}_{r_{11}}, \ldots, X^{1n_1}_{r_{1n_1}} \rangle \to Y^1$$

$\ldots$

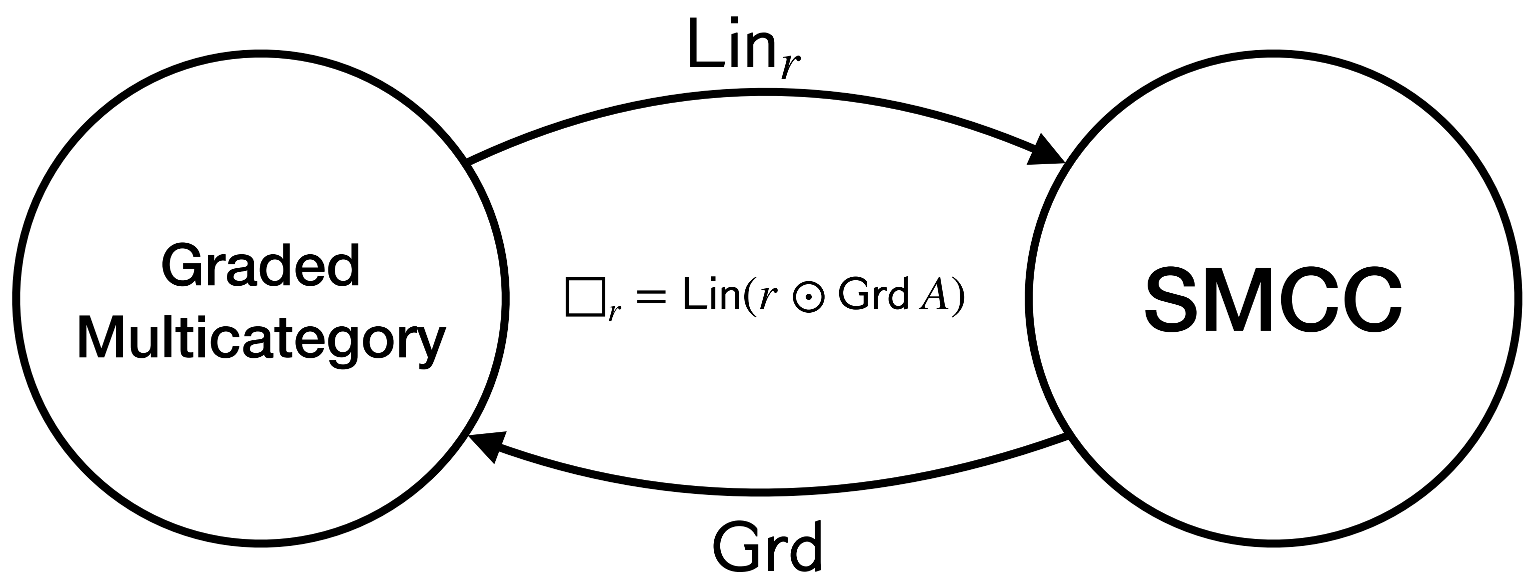
$$f_m : \langle X^{m1}_{r_{m1}}, \ldots, X^{mn_m}_{r_{mn_m}} \rangle \to Y^m$$

$$g : \langle Y^1_{s_1}, \ldots, Y^m_{s_m} \rangle \to Z$$

then:

$$g(f_1, \ldots, f_m) : \langle X^{11}_{s_1 \circledast r_{11}}, \ldots, X^{1n_1}_{s_1 \circledast r_{1n_1}}, \ldots, X^{m1}_{s_m \circledast r_{m1}}, \ldots, X^{mn_m}_{s_m \circledast r_{mn_m}} \rangle \to Z$$

Mixed Graded/Linear Logic



$$r \odot A_s = A_{r \circledast s} : \mathscr{R} \times \mathscr{G} \to \mathscr{G}$$

Graded Logic

$$\frac{(\phi_1, \phi_2) \odot (\Phi_1, \Phi_2) \vdash_{\mathscr{G}} Y}{(\phi_1, \mathsf{a}, \phi_2) \odot (\Phi_1, X, \Phi_2) \vdash_{\mathscr{G}} Y}\ \textbf{weak}$$

$$\frac{\phi_2 \odot \Phi_2 \vdash_{\mathscr{G}} X \qquad (\phi_1, r, \phi_3) \odot (\Phi_1, X, \Phi_3) \vdash_{\mathscr{G}} Y}{(\phi_1, r \circledast \phi_2, \phi_3) \odot (\Phi_1, \Phi_2, \Phi_3) \vdash_{\mathscr{G}} Y}\ \textbf{cut}$$

$$\frac{(\phi_1, r_1, r_2, \phi_2) \odot (\Phi_1, X, X, \Phi_2) \vdash_{\mathscr{G}} Y}{(\phi_1, (r_1 \oplus r_2), \phi_2) \odot (\Phi_1, X, \Phi_2) \vdash_{\mathscr{G}} Y}\ \textbf{cont}$$

Mixed Graded/Linear Logic

$$\frac{(\phi, r) \odot (\Delta, X); \Gamma \vdash_{\mathscr{M}} C}{\phi \odot \Delta; (\mathsf{Lin}_r X, \Gamma) \vdash_{\mathscr{M}} C}\ \textbf{LinL}$$

$$\frac{\phi \odot \Delta \vdash_{\mathscr{G}} X}{(r \circledast \phi) \odot \Delta; \varnothing \vdash_{\mathscr{M}} \mathsf{Lin}_r X}\ \textbf{LinR}$$

Graded Adjoint Logic

Adjoint Logic:
Modes control which structural rules you have.

Grades Types:
Grades control how you use structural rules.

But, what's a mode anyway?

Graded Adjoint Logic

Adjoint Logic:

Modes control which structural rules you have.

Grades Types:

Grades control how you use structural rules.

Modes! Modes! I think you mean, graded modes!

Graded Adjoint Logic : Partiality

Semiring : $(\mathscr{R}, \mathsf{m}, \circledast, \mathsf{a}, \oplus, \leq, \mathsf{Weak}, \mathsf{Cont}, \mathsf{Comp})$

where $\mathsf{Weak} \subseteq \mathscr{R}$

$\mathsf{Cont} \subseteq \mathscr{R} \times \mathscr{R} \times \mathscr{R}$

$\mathsf{Comp} \subseteq \mathscr{R} \times \mathscr{R} \times \mathscr{R}$

Judgments : $\phi \odot \Phi \vdash_{\mathscr{G}} X$

$\phi \odot \Phi; \Gamma \vdash_{\mathscr{M}} A$

Modalities : $\mathsf{Lin}_r\, X$

$\mathsf{Grd}\, A$

Graded Adjoint Logic : Partiality

$$\frac{r \in \mathsf{Weak} \quad (\phi_1, \phi_2) \odot (\Phi_1, \Phi_2) \vdash_{\mathscr{G}} Y}{(\phi_1, \mathsf{a}, \phi_2) \odot (\Phi_1, X, \Phi_2) \vdash_{\mathscr{G}} Y} \ \textbf{weak}$$

$$\frac{(r_1 \oplus r_2) \in \mathsf{Cont} \quad (\phi_1, r_1, r_2, \phi_2) \odot (\Phi_1, X, X, \Phi_2) \vdash_{\mathscr{G}} Y}{(\phi_1, (r_1 \oplus r_2), \phi_2) \odot (\Phi_1, X, \Phi_2) \vdash_{\mathscr{G}} Y} \ \textbf{cont}$$

Graded Adjoint Logic : Partiality

$(\{l, w, c\}, l, \circledast, w, \oplus)$

r_1	l	l	w	c	l	w	c	w	c
r_2	w	c	l	l	l	w	c	c	w
$r_1 \oplus r_2$	$*$	$*$	$*$	$*$	$*$	$*$	c	$*$	$*$

Graded Adjoint Logic

Graded modes are partial semirings.

Graded Adjoint Logic

Graded modes are partial semirings.

How do we get more than one mode?

How do we relate them through adjunctions?

Graded Adjoint Logic

Given

$$(\mathscr{R}_1, \mathsf{m}_1, \circledast_1, \mathsf{a}_1, \oplus_1, \leq_1, \mathsf{Weak}_1, \mathsf{Cont}_1, \mathsf{Comp}_1)$$

$$(\mathscr{R}_2, \mathsf{m}_2, \circledast_2, \mathsf{a}_2, \oplus_2, \leq_2, \mathsf{Weak}_2, \mathsf{Cont}_2, \mathsf{Comp}_2)$$

and

$$g : \mathscr{R}_1 \to \mathscr{R}_2$$

$h : \mathscr{R}_2 \to \mathscr{R}_1$ **where** $g(r_1) \leq_2 r_2 \iff r_1 \leq_1 h(r_2)$

Graded Adjoint Logic

Given

$$(\mathscr{R}_1, \mathsf{m}_1, \circledast_1, \mathsf{a}_1, \oplus_1, \leq_1, \mathsf{Weak}_1, \mathsf{Cont}_1, \mathsf{Comp}_1)$$

$$(\mathscr{R}_2, \mathsf{m}_2, \circledast_2, \mathsf{a}_2, \oplus_2, \leq_2, \mathsf{Weak}_2, \mathsf{Cont}_2, \mathsf{Comp}_2)$$

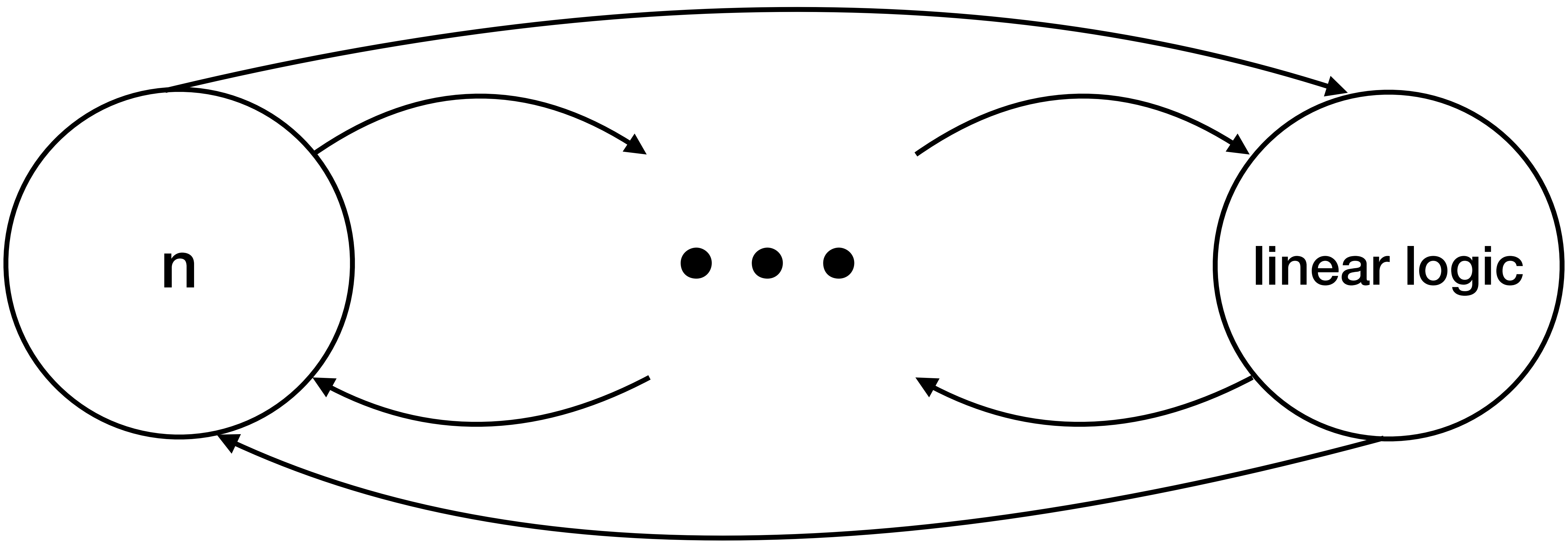
and

$$g : \mathscr{R}_1 \to \mathscr{R}_2$$

$h : \mathscr{R}_2 \to \mathscr{R}_1$ **where** $g(r_1) \leq_2 r_2 \iff r_1 \leq_1 h(r_2)$

These imply an ordering on modes!

Families of Modalities through Adjunctions



Graded Adjoint Logic

Given

$$(\mathscr{R}_1, \mathsf{m}_1, \circledast_1, \mathsf{a}_1, \oplus_1, \leq_1, \mathsf{Weak}_1, \mathsf{Cont}_1, \mathsf{Comp}_1)$$

$$(\mathscr{R}_2, \mathsf{m}_2, \circledast_2, \mathsf{a}_2, \oplus_2, \leq_2, \mathsf{Weak}_2, \mathsf{Cont}_2, \mathsf{Comp}_2)$$

and

$g : \mathscr{R}_1 \to \mathscr{R}_2$

$h : \mathscr{R}_2 \to \mathscr{R}_1$ **where** $g(r_1) \leq_2 r_2 \iff r_1 \leq_1 h(r_2)$

$$\mathscr{R}_1 \geq \mathscr{R}_2 = \{(g, h) \mid \mathscr{R}_1 : g \dashv h : \mathscr{R}_2\}$$

Graded Adjoint Logic

$$(\{\mathsf{w},\mathsf{l}\},\mathsf{l},\circledast_1,\mathsf{w},\oplus_1,\{\mathsf{w}\leq_1\mathsf{l}\},\{\mathsf{w}\},\varnothing,\{\mathsf{w},\mathsf{l}\})$$

$$(\{\mathsf{c},\mathsf{w},\mathsf{l}\},\mathsf{l},\circledast_2,\mathsf{w},\oplus_2,\{\mathsf{w}\leq_1\mathsf{l},\mathsf{c}\leq_2\mathsf{l}\},\{\mathsf{w}\},\{\mathsf{c}\},\{\mathsf{c},\mathsf{w},\mathsf{l}\})$$

$$g : \{\mathsf{c},\mathsf{w},\mathsf{l}\} \to \{\mathsf{w},\mathsf{l}\}$$
$$g(w) = w$$
$$g(l) = l$$
$$g(c) = l$$

$$h : \{\mathsf{w},\mathsf{l}\} \to \{\mathsf{c},\mathsf{w},\mathsf{l}\}$$
$$h(w) = w$$
$$h(l) = l$$

$$(g(\mathsf{c}) \leq_1 \mathsf{l} \iff \mathsf{c} \leq_2 h(\mathsf{l})) \iff (\mathsf{l} \leq_1 \mathsf{l} \iff \mathsf{c} \leq_2 \mathsf{l})$$

Graded Adjoint Logic

Suppose:

$\mathscr{M}$ **is a SMC**

$(\mathscr{R}_1, \mathsf{m}_1, \circledast_1, \mathsf{a}_1, \oplus_1, \leq_1, \mathsf{Weak}_1, \mathsf{Cont}_1, \mathsf{Comp}_1)$

$(\mathscr{R}_2, \mathsf{m}_2, \circledast_2, \mathsf{a}_2, \oplus_2, \leq_2, \mathsf{Weak}_2, \mathsf{Cont}_2, \mathsf{Comp}_2)$

$\mathscr{R}_1 \geq \mathscr{R}_2 = \{(g, h) \mid \mathscr{R}_1 : g \dashv h : \mathscr{R}_2\}$

Then:

$$\mathsf{Gr}(\mathscr{R}_1, \mathscr{M}) : G \dashv H : \mathsf{Gr}(\mathscr{R}_2, \mathscr{M})$$

Graded Adjoint Logic

$$\frac{}{(\mathsf{mid}_m : \mathsf{m}) \odot A \vdash_m A}\ \textbf{Id}$$

Graded Adjoint Logic

$$\frac{(\phi_1, r : m_1, r : m_1, \phi_2) \odot (\Delta_1, A, B, \Delta_2) \vdash_{m_2} C}{(\phi_1, r : m_1, \phi_2) \odot (\Delta_1, A \otimes B, \Delta_2) \vdash_{\mathsf{m}_2} C}\ \textbf{TenL}$$

Graded Adjoint Logic

$$\frac{\phi_2 \geq m_1 \quad \phi_2 \odot \Delta_2 \vdash_{m_1} A \quad (\phi_1, r : m_1, \phi_3) \odot (\Delta_1, A, \Delta_3) \vdash_{m_2} C}{(\phi_1, r \circledast_{m_1} \phi_2, \phi_3) \odot (\Delta_1, \Delta_2, \Delta_3) \vdash_{m_2} C} \text{Cut}$$

Graded Adjoint Logic

$$\frac{m_2 \geq m_1 \qquad \phi \odot \Delta \vdash_{m_1} A}{\phi \odot \Delta \vdash_{m_2} \uparrow_{m_1}^{m_2} A}\ \textbf{Up}$$

$$\frac{\begin{array}{l} m_2 \geq m_1 \\ \vdash r \circledast_{m_2} \phi \end{array} \qquad \phi \odot \Delta \vdash_{m_2} A}{(r \circledast \phi) \odot \Delta \vdash_{m_1} \downarrow_{m_1}^{m_2} A}\ \textbf{Down}$$

Take aways!

- **Adjoint Logic : which structural rules can be used.**
- **Graded Logic : how structural rules can be used.**
- **Graded Adjoint Logic : Combining the best of both worlds.**
- **Granule Project : https://granule-project.github.io/**
- **Find me:**
  - **@heades on Twitter**
  - **metatheorem.org**