Skolemization in Nonclassical Logic

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Skolemization

The connection between quantifier combination $\forall\exists$ and functions appears at many places:

- axiom of choice:
  $\forall x \in a \exists y \in x \rightarrow \exists(f : a \rightarrow \bigcup a) \forall x \in a(fx \in x).$

- constructive mathematics: (for quantifier-free φ)
  if $HA \vdash \forall x \exists y \varphi(x, y)$, then $HA \vdash \forall x \varphi(x, fx)$ for some computable f.

  $\vdots$

Constructive interpretation of quantifiers:

$$\vdash \forall x \in A \exists y \in B \, \varphi(x, y)$$
$$\Leftrightarrow$$
$$\vdash \forall x \in A \, \varphi(x, fx) \text{ for some function } f : A \rightarrow B.$$

The study of the constructive content of such quantifier combinations is pursued in contructive mathematics and proof mining.

In classical logic the following holds for the dual quantifier combination:

$$\vdash \exists x \in A \forall y \in B \, \varphi(x, y)$$
$$\Leftrightarrow$$
$$\vdash \exists x \in A \, \varphi(x, fx) \text{ for some function } f : A \to B \text{ not in } \varphi.$$

It can be understood via countermodels:

$$M \vDash \forall x \in A \exists y \in B \, \neg\varphi(x, y) \text{ for some model } M$$
$$\Leftrightarrow$$
$$N \vDash \forall x \in A \neg\varphi(x, fx) \text{ for some model } N \text{ and function } f : A \to B \text{ not in } \varphi.$$

Or via satisfiability:

$$\forall x \in A \exists y \in B \, \neg\varphi(x, y) \text{ is satisfiable}$$
$$\Leftrightarrow$$
$$\forall x \in A \neg\varphi(x, fx) \text{ is satisfiable for some function } f : A \to B \text{ not in } \varphi.$$

The dual quantifier combination $\exists\forall$ appears in the skolemization method for theories in classical predicate logic CQC.

Thm For any function symbols f, g not in φ ($\vdash$ short for $\vdash_{\mathrm{CQC}}$, fx for $f(x)$):

$$\vdash \exists x\forall y\varphi(x, y) \quad \Leftrightarrow \quad \vdash \exists x\varphi(x, fx)$$
$$\vdash \exists x\forall y\exists u\forall v\varphi(x, y, u, v) \quad \Leftrightarrow \quad \vdash \exists x\exists u\varphi(x, f(x), u, g(x, u))$$
$$\vdash \exists x_1\forall y_1 \ldots \exists x_n\forall y_n\varphi(\bar{x}, \bar{y}) \quad \Leftrightarrow \quad \vdash \exists x_1 \ldots \exists x_n\varphi\big(\bar{x}, f_1(x_1), \ldots, f_n(x_1, \ldots, x_n)\big).$$

This talk: Skolemization in intermediate logics.

In CQC:

$$\vdash_{\mathrm{CQC}} \neg \exists x \forall y \varphi(x, y) \Leftrightarrow$$
$$\vdash_{\mathrm{CQC}} \forall x \exists y \neg \varphi(x, fx) \Leftrightarrow$$
$$\vdash_{\mathrm{CQC}} \forall x \neg \varphi(x, fx) \Leftrightarrow$$
$$\vdash_{\mathrm{CQC}} \neg \exists x \varphi(x, fx).$$

In intuitionistic logic IQC formulas do not have a prenex normal form.

Thus skolemization needs to be extended to infix formulas.

(Thoralf Skolem 1887–1963)

Def The *strong* quantifier occurrences are positive occurrences of $\forall$ and negative occurrences of $\exists$. All other quantifier occurrences are *weak*.

Strong quantifiers become universal under prenexification, and weak quantifiers become existential.

Ex Strong quantifier occurrences are red and weak occurrences are green:

$$\forall x\varphi(x) \lor \exists y\psi(y) \to \forall x\psi(x) \lor \exists y\varphi(y)$$

$$(\forall x\varphi(x) \to \forall y\psi(y)) \to \neg\neg\forall x\psi(x) \land \neg\exists y\varphi(y)$$

The skolemization of infix formulas replaces the strong quantifiers on the spot (def next slide).

Ex

	formula	skolemization
	$\forall x\varphi(x) \to \forall x\varphi(x)$	$\forall x\varphi(x) \to \varphi(c)$
	$\exists x(\exists u\varphi(x,u) \to \exists y\forall v\psi(x,v))$	$\exists x(\varphi(x,fx) \to \exists y\psi(x,g(x,y))).$

Def The *skolemization* φ^s of φ is the result of replacing (from inside out) the occurrences of strong quantifiers $Qy\psi(\bar{x}, y)$ in φ by $\psi(\bar{x}, f(\bar{x}))$, where f is fresh and $\bar{x}$ are the variables of the weak quantifiers in which scope $Qy\psi(\bar{x}, y)$ occurs.

A logic L *admits* skolemization if for all formulas φ: $\vdash_{\mathrm{L}} \varphi$ if and only if $\vdash_{\mathrm{L}} \varphi^s$

Thm CQC admits skolemization: $\vdash_{\mathrm{CQC}} \varphi$ if and only if $\vdash_{\mathrm{CQC}} \varphi^s$.

Def The *skolem Class* of a theory T: $SC(\mathrm{T}) \equiv_{df} \{\varphi \mid \vdash_{\mathrm{T}} \varphi \Leftrightarrow \vdash_{\mathrm{T}} \varphi^s\}$.

Equivalently:

$$SC(\mathrm{T}) = \{\varphi \mid \vdash_{\mathrm{T}} \varphi \text{ and } \vdash_{\mathrm{T}} \varphi^s, \text{ or } \nvdash_{\mathrm{T}} \varphi \text{ and } \nvdash_{\mathrm{T}} \varphi^s, \}.$$

Note SC(CQC) consists of all formulas. SC(IQC) does not.

In CQC we have:

Thm $\vdash \varphi \Leftrightarrow \vdash \varphi^s$.

Thm (Herbrand's Theorem) $\vdash \exists x \varphi(x) \Leftrightarrow \vdash \bigvee_{i=1}^{n} \varphi(t_i)$ for some terms t_i.

In combination: $\vdash \exists x \forall y \varphi(x, y) \Leftrightarrow \vdash \bigvee_{i=1}^{k} \varphi(x, f(t_i))$ for some terms t_i.

Likewise for longer prefixes of quantifiers.

Applications: Computational content of proofs, automated theorem proving, connection propositional and predicate logic.

Question: How about other intermediate predicate logics/theories?

Def An intermediate predicate logic/theory is a logic/theory that is an extension of intuitionistic predicate logic IQC.

Nonclassical theories

For many intermediate predicate theories T no skolemization theorem:

$$\vdash_{\mathrm{T}} \varphi \;\Rightarrow\; \vdash_{\mathrm{T}} \varphi^s \quad \vdash_{\mathrm{T}} \varphi \not\Leftarrow \vdash_{\mathrm{T}} \varphi^s.$$

Ex The CD formula in IQC:

$$\not\vdash_{\mathrm{IQC}} \forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi \quad \vdash_{\mathrm{IQC}} \forall x(\varphi(x) \vee \psi) \to \varphi(c) \vee \psi$$

Thus $\mathrm{CD} \notin SC(\mathrm{IQC})$.

In many intermediate predicate logics the (generalized) Herbrand theorem holds:

For any formula φ with only weak quantifiers there exists an Herbrand expansion φ^H of φ that is quantifier free and $\vdash_{\mathrm{T}} \varphi \;\Leftrightarrow\; \vdash_{\mathrm{T}} \varphi^H$.

- For which (intermediate) logics is skolemization complete?
- What is the skolem Class of a given logic?

Mostly partial answers.

Results

Last two decades: various results on skolemization in nonclassical logics:

- A sufficient condition on formulas for belonging to the skolem Class of IQC. (Mints 1970s)
- The prenex fragment belongs to the skolem Class of wide range of first-order fuzzy logics. (Baaz & Ciabattoni & Fermüller 2001), (Baaz & Metcalfe 2010)
- First-order Łukasiewicz logic admits skolemization.
  (Baaz & Metcalfe 2010)

  Certain formula classes belong to the the skolem Class of first-order substructural logics. (Cintula & Metcalfe 2013)
- IQCE admits an alternative skolemization method.
  (Baaz & Iemhoff 2006).
- There is a labelled version of IQC that admits skolemization.
  (Baaz & Iemhoff 2008).
- and many many more …

During a visit to Matthias Baaz in 2017:
All prenex formulas are in the skolem Class of IQC. What about other intermediate logics?

Thm For any well-founded tree-complete intermediate logic, any prenex formula belongs to the skolem Class of the logic.

Proof Nontrivial.

Question: For which propositional formulas $A(p_1, \ldots, p_n)$ does $A(\varphi_1, \ldots, \varphi_n)$ belong to the skolem Class of a logic for any prenex formulas $\varphi_1, \ldots, \varphi_n$?

Ex For IQC:

Positive answer: p_1 and $p_1 \wedge p_2$.

Negative anwer: $p_1 \rightarrow \neg\neg p_2$
$\exists x \neg\neg\varphi(x) \rightarrow \neg\neg \exists x \varphi(x)$ not in $SC(\mathrm{IQC})$ for quantifier free φ.

Negative answer: $p_1 \rightarrow p_2 \vee p_3$
$\forall x(\varphi(x) \vee \psi) \rightarrow (\forall x \varphi(x) \vee \psi)$ not in $SC(\mathrm{IQC})$ for quantifier free φ, ψ.

Note A, B, C for propositional formulas, φ, ψ, χ for predicate formulas.

Def An intermediate logic is *well-founded tree-complete* if it is complete with respect to a class of well-founded trees.

Ex Any intermediate logics with the finite frame property is well-founded tree-complete. IQC, KC, LC are examples.

Def A is *rigid*, i.e. no atom occurs twice in A. A is a *nni* formula: no implication occurs negatively in it.

Ex $\bigvee_i \bigwedge_j (\bigwedge_k p_{ijk} \to \bigvee_l q_{ijl})$ is rigid.

Def $\varphi_1, \ldots, \varphi_n$ independent if no predicate occurs in more than one φ_i.

Thm (I. 2018) In any well-founded tree-complete intermediate logic L, for any rigid nni formula $A(p_1, \ldots, p_n)$, for all independent prenex formulas $\varphi_1, \ldots, \varphi_n$: $A(\varphi_1, \ldots, \varphi_n) \in SC(\mathrm{L})$.

The proof uses skolemization and its dual, i.e. skolemization for derivability and satisfiability.

Thm (I. 2018) In any well-founded tree-complete intermediate logic L, for any rigid nni formula $A(p_1, \ldots, p_n)$, for all independent prenex formulas $\varphi_1, \ldots, \varphi_n$: $A(\varphi_1, \ldots, \varphi_n) \in SC(\mathrm{L})$.

Ex For all independent $\varphi_1, \ldots, \varphi_n$: $(\varphi_1 \wedge \varphi_2 \to \varphi_3 \vee \varphi_4) \in SC(\mathrm{IQC})$.
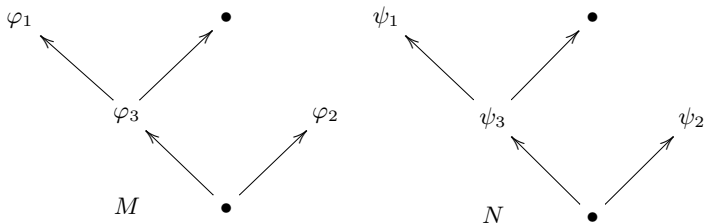
Note Recall that

$$\forall x(\varphi(x) \vee \psi) \to \forall x \varphi(x) \vee \psi \notin SC(\mathrm{IQC}).$$

That $p_1 \to p_2 \vee p_3$ is a rigid nni formula does not contradict the theorem, as $\forall x(\varphi(x) \vee \psi), \forall x \varphi(x), \psi$ are not independent.

Whether the theorem holds for $A(p_1, p_2, p_3) = (p_1 \to p_2 \vee p_3)$: open.

Proof Let $\big(A(\varphi_1, \ldots, \varphi_n)\big)^s = A(\psi_1, \ldots, \psi_n)$. Given $M \nVdash A(\varphi_1, \ldots, \varphi_n)$, create $N \nVdash \big(A(\varphi_1, \ldots, \varphi_n)\big)^s$.

$$\psi_i = \begin{cases} (\varphi^i)^s & \text{if } p_i \text{ occurs positively in } A \\ (\overline{\varphi}^i)^s & \text{if } p_i \text{ occurs negatively in } A, \end{cases}$$

$\overline{\varphi}$ results from swapping the quantifiers of φ.

We have seen some partial answers (from the literature and this talk) to the questions:

- For which (intermediate) logics is skolemization complete?
- What is the skolem Class of a given logic?

Alternative methods

Def An *alternative skolemization method* is a computable total translation $(\cdot)^a$ from formulas to formulas such that for all formulas φ, φ^a does not contain strong quantifiers. A logic L *admits* the alternative skolemization method if

$$\vdash_{\mathrm{L}} \varphi \iff \vdash_{\mathrm{L}} \varphi^a. \tag{1}$$

The method is *strict* if for all Kripke models K of L and all formulas φ:

$$K \not\Vdash \varphi^a \implies K \not\Vdash \varphi. \tag{2}$$

Motivation: Alternative skolemization methods preserve the connection between predicate and propositional logic in combination with Herbrand's theorem.

Note In many intermediate predicate logics the (generalized) Herbrand theorem holds: For any formula φ with only weak quantifiers there exists an Herbrand expansion φ^H of φ that is quantifier free and $\vdash_{\mathrm{T}} \varphi \iff \vdash_{\mathrm{T}} \varphi^H$.

Def A logic has *width n* if it is complete with resprect to a class of models that have no anti-chains of lentgh $> n$.

Def Given a number n, *parallel skolemization* replaces strong quantifiers $\exists x\psi(x,\bar{y})$ and $\forall x\psi(x,\bar{y})$ by, respectively,

$$\bigvee_{i=1}^{n} \psi(f_i(\bar{y}),\bar{y}) \text{ and } \bigwedge_{i=1}^{n} \psi(f_i(\bar{y}),\bar{y}).$$

Thm (Baaz&Iemhoff 2016) Any intermediate logic of finite width with constant domains admits parallel skolemization.

Cor Any tabular constant domain logic admits parallel skolemization.

From later work: it probably holds for intermediate logics with the fmp and constant domains.

Thm (I. 2017) Except for CQC, there is no Kripke complete intermediate logic that admits a strict alternative skolemization method.

Cor The intermediate logics IQC,

- QDn (the logic of frames of branching at most n),
- QKC (the logic of frames with one maximal node),
- QLC (the logic of linear frames),

and all tabular logics, do not admit any strict, alternative skolemization method.

Def The *strong quantifier free fragment (sqff)* of a logic consists of those theorems of the logic that do not contain strong quantifiers, and likewise for weak quantifiers.

Thm (I. 2017) Except for CQC, there is no intermediate logic that is complete with respect to a class of frames and admits a strict alternative skolemization method.

Proof idea Let L be an intermediate logic that is complete with respect to a class of frames and let $\mathcal{K}$ be the class of models based on these frames. Let $(\cdot)^a$ be the alternative skolemization method.

Claim L is sound and complete with respect to $\mathcal{K}_{cd}$ the class of all models in $\mathcal{K}$ with constant domain.

Proof If $\nvdash \varphi$, then $\nvdash \varphi^a$. Thus $K \nVdash \varphi^a$ for some $K \in \mathcal{K}$. Because φ^a no strong qfs, $K^\downarrow \nVdash \varphi^a$, which is K in which every domain is replaced by that at the root.

$K^\downarrow \in \mathcal{K}_{cd}$, which proves the claim. ⊣

So L derives the constant domain formula (CD) $\forall x(\varphi(x) \vee \psi) \rightarrow \forall x\varphi(x) \vee \psi$.

As $L \neq CQC$, there is at least one frame of L of depth > 1. On such a frame CD can be refuted. A contradiction.

⊣

- Different work skolemization e.g. Avigad 2003 on shortening proofs.
- With Raheleh Jalali: connection between quantifier shifts and skolemization.
- Do requirements on skolem functions lead to more logics admitting the method?
- Are there useful alternatives to alternative skolemization methods?

⋮

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